Self-healing pulse solution in a continuum model of fracture propagation

Hiizu Nakanishi
Department of Physics, Kyushu University 33, Fukuoka 812-8581, Japan
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An analytical solution that represents a self-healing pulse of slip is presented for a dynamical model of fracture in a two-dimensional continuum medium. Even without the cohesive region, the solution does not show a singular behavior in the stress at the resticking point unlike at the breaking point, where the stress is diverging as $1/\sqrt{r}$. This means that the physical condition at the resticking point should depend on the microscopic processes of resticking while the condition at the breaking point is known to be described by the phenomenological fracture energy.

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It requires a large amount of work to slide a carpet on a floor by pulling one of its sides because the friction force is acting over the whole interface between the carpet and the floor. But, if one makes a bump at one side and pushes it through to the opposite side, the carpet can be shifted much easier because only a small fraction of the carpet is moving at a given time and the friction force to the bump is small. A more microscopic example that is familiar to physicists is a dislocation propagation; a crystal deforms by propagating dislocations, thus it can be deformed much easier than it would be expected if all the bonds would have to be shifted at the same time.

It has been realized that a similar thing is happening in much larger scale; An earthquake fault does not slip at once in a big event, but it slides as a localized pulse of fracture propagates along the fault, and the fault surfaces stick together after the fracture front has passed. From the analysis of seismic data, it has been found that the length of slipping region of the earthquake fault at a given moment within an event is much shorter than the total length of the fault broken by the earthquake [1]. Simulations of a simple block-spring model also show that fracture fronts propagate as a narrow pulse in a big sliding event [3–5].

On the other hand, theoretical study on fracture dynamics is largely limited to the fracture without healing and little is known about physical and mathematical properties of the pulse such as crack propagation with self-healing. There has been found a solution for a pulse propagation in a two-dimensional continuum [2], but the solution is not a dynamic but a kinematic one in the sense that its dynamical features such as the fracture speed and the pulse length are arbitrary and cannot be determined for a given physical situation. In this paper, I will solve a dynamical model of fracture propagation and present a pulse solution of the mode III crack with self-healing in a two-dimensional continuum.

The model I study here is the same with the one where the analytical solution without self-healing has been obtained [6–8]. I consider only the out of plane component (z component) of the displacement field $u(x,y,t)$, which satisfies the equation of motion

$$\ddot{u}(x,y,t)=c_0^2\nabla^2u(x,y,t)-\omega_0^2[u(x,y,t)-\Delta] \quad (y>0), \tag{1}$$

where c_0 is the sound speed for the material, Δ is the external displacement imposed at infinity ($y\to\infty$), and ω_0 is the angular frequency around it and represents the stiffness of the externally imposed displacement; in the lower half plane ($y<0$), we suppose that the external displacement is $-\Delta$. The crack is supposed to propagate along the x axis toward the $-x$ direction. We assume the anti-symmetric situation for the displacement u about the x axis and consider only the upper half plane (Fig. 1).

Suppose the crack tip is moving at the constant velocity $-v$ and its position is $-vt$, then the boundary condition along the x axis is

$$\mu\frac{\partial u}{\partial y}\Big|_{y=0}=\begin{cases}\sigma^{(-)}(x) & (x<-vt),\\ \sigma^{(+)}(x) & (x>-vt),\end{cases} \tag{2}$$

$$u(x,y=+0)=\begin{cases}0 & (x<-vt),\\ U(x) & (x>-vt),\end{cases} \tag{3}$$

where $\sigma^{(+)}(x)$ $[\sigma^{(-)}(x)]$ is the stress along the x axis for $y>-vt$ $(y<-vt)$, and $U(x)$ is half of the crack opening; μ denotes the elastic modulus.

For convenience we define the width of the distorted region W, the strain ε_∞, and the stress σ_∞ at the center line at $x=-\infty$ by

$$W\equiv\frac{c_0}{\omega_0}, \quad \varepsilon_\infty\equiv\frac{\Delta}{W}, \quad \sigma_\infty\equiv\mu\varepsilon_\infty . \tag{4}$$

σ_∞ represents the stress imposed by the external distortion.

Now we transform the coordinate system into the moving frame where the crack tip is located at the origin as $x+vt$

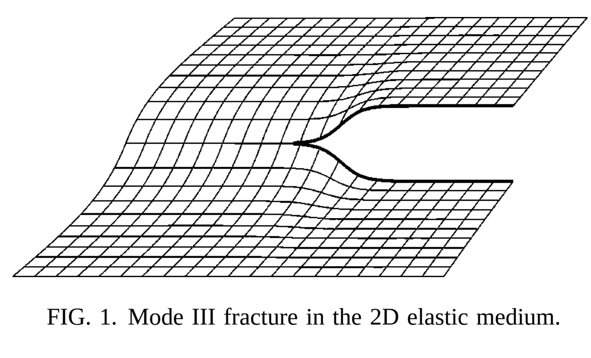

FIG. 1. Mode III fracture in the 2D elastic medium.

$\to x$, and look for the stationary solution by replacing $\partial/\partial t \to v\,\partial/\partial x$, then the equation of motion for $y>0$ becomes

$$c_0^2\left[\left(1-\frac{v^2}{c_0^2}\right)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]u(x,y)-\omega_0^2[u(x,y)-\Delta]=0. \tag{5}$$

The solution with the appropriate boundary condition can be expressed as [6,7]

$$u(x,y)=\Delta(1-e^{-y/W})+\int\frac{dk}{2\pi}\hat{U}^{(+)}(k)e^{+ikx-\hat{K}(k)y} \tag{6}$$

with

$$\hat{K}(k)\equiv\beta\sqrt{\alpha^2+k^2},\quad \alpha\equiv\frac{1}{W\beta},\quad \beta\equiv\sqrt{1-v^2/c_0^2}, \tag{7}$$

where $\hat{U}^{(+)}(k)$ is the Fourier transform of $U(x)$; the suffix $(+)$ denotes that all the singularities of $\hat{U}^{(+)}(k)$ are located in the upper half plane because $U(x)=0$ for $x<0$.

This can be solved explicitly for $U(x)$ and $\sigma^{(-)}(x)$. In the $\alpha\to 0$ limit, or the infinite system width limit, the solution can be expressed in the simple form in the real space representation [9]

$$U'(x)=\frac{1}{\mu\beta}\mathrm{P}\int_0^\infty\frac{dy}{\pi}\sqrt{\frac{x}{y}}\frac{1}{x-y}\sigma^{(+)}(y)\quad\text{for}\ (x>0), \tag{8}$$

$$\sigma^{(-)}(x)=\mathrm{P}\int_0^\infty\frac{dy}{\pi}\sqrt{\frac{-x}{y}}\frac{1}{y-x}\sigma^{(+)}(y)\quad\text{for}\ (x<0) \tag{9}$$

in terms of the stress behind the crack tip $\sigma^{(+)}(x)$; $P\int$ denotes the Cauchy principal value of the integral. The condition that the stress should not diverge at the crack tip is shown to be

$$\frac{1}{\sqrt{\pi}}\int_0^\infty dx\frac{e^{-\alpha x}}{\sqrt{x}}[\sigma^{(+)}(x)-\sigma_\infty]=0. \tag{10}$$

Note that the expression for the general case, $\alpha\neq 0$, is given for this.

Now we look for a self-healing pulse solution. We assume the crack surfaces restick at $x=L$, then the stress behind the crack tip $\sigma^{(+)}(x)$ consists of two parts, namely, the slipping stress $\sigma_{\mathrm{sl}}(x)$ for $0<x<L$, and the resticking stress $\sigma_{\mathrm{st}}(x)$ for $L<x$. We assume the free traction condition for the slipping stress except for the cohesive zone where the simple constant cohesive stress σ_y operates within the cohesive range $U(x)<\delta$ (Fig. 2)

$$\sigma_{\mathrm{sl}}(x)=\sigma_c[U(x)]\quad\text{for}\ (0<x<L) \tag{11}$$

and

$$\sigma_c(U)=\begin{cases}\sigma_y & (0<U<\delta),\\ 0 & (U>\delta).\end{cases} \tag{12}$$

If we denote the length of the cohesive region ℓ, then

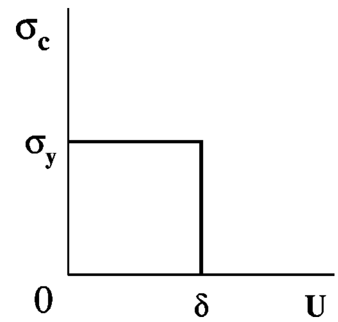

FIG. 2. Simple model of the cohesive stress.

$$U(\ell)=\delta \tag{13}$$

and

$$\sigma_{\mathrm{sl}}(x)=\begin{cases}\sigma_y & (0<x<\ell),\\ 0 & (\ell<x<L),\end{cases} \tag{14}$$

therefore Eq. (8) becomes

$$U'(x)=\frac{1}{\mu\beta}\mathrm{P}\int_0^\ell\frac{dy}{\pi}\sqrt{\frac{x}{y}}\frac{1}{x-y}\sigma_y+\frac{1}{\mu\beta}\mathrm{P}\int_L^\infty\frac{dy}{\pi}\sqrt{\frac{x}{y}}\frac{1}{x-y}\sigma_{\mathrm{st}}(y). \tag{15}$$

Since the crack opening $U(x)$ is constant in the resticking region, Eq. (15) becomes zero for $x>L$:

$$0=U_0'(x)+\frac{1}{\mu\beta}\mathrm{P}\int_L^\infty\frac{dy}{\pi}\sqrt{\frac{x}{y}}\frac{1}{x-y}\sigma_{\mathrm{st}}(y)\quad\text{for}\ (x>L), \tag{16}$$

where we have defined the crack opening $U_0(x)$ in the case without resticking by

$$U_0'(x)\equiv\frac{1}{\mu\beta}\mathrm{P}\int_0^\ell\frac{dy}{\pi}\sqrt{\frac{x}{y}}\frac{1}{x-y}\sigma_y=\frac{\sigma_y}{\pi\mu\beta}\ln\left|\frac{\sqrt{\ell}+\sqrt{x}}{\sqrt{\ell}-\sqrt{x}}\right|. \tag{17}$$

The singular integral equation (16) can be solved for $\sigma_{\mathrm{st}}(x)$ (see the Appendix and Ref. [10]) and then we obtain the expressions for $U'(x)$ and $\sigma^{(-)}(x)$ from Eqs. (15) and (9), respectively,

$$\sigma_{\mathrm{st}}(x)=-\mu\beta\,\mathrm{P}\int_L^\infty\frac{dy}{\pi}\frac{1}{y-x}\sqrt{\frac{x}{y}}\sqrt{\frac{x-L}{y-L}}U_0'(y)\quad\text{for}\ (x>L), \tag{18}$$

$$U'(x)=U_0'(x)-\int_L^\infty\frac{dy}{\pi}\frac{1}{y-x}\sqrt{\frac{x}{y}}\sqrt{\frac{L-x}{y-L}}U_0'(y)\quad\text{for}\ (0<x<L), \tag{19}$$

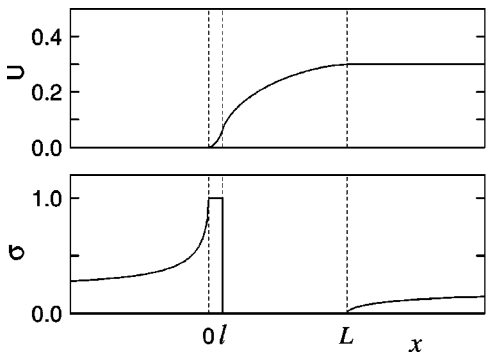

FIG. 3. $U(x)$ and $\sigma(x)$ vs x for the case $\ell/L=0.1$. $U(x)$ is given in the unit of $\sigma_y/\mu\beta$ and $\sigma(x)$ in the unit of σ_y.

$$\sigma^{(-)}(x)=\sigma_0^{(-)}(x)+\mu\beta\int_L^{\infty}\frac{dy}{\pi}\frac{1}{y-x}\times\sqrt{\frac{-x}{y}}\sqrt{\frac{L-x}{y-L}}U_0'(y)\quad\text{for } (x<0), \tag{20}$$

where $\sigma_0^{(-)}(x)$ is the stress for the case without resticking;

$$\sigma_0^{(-)}(x)\equiv\frac{2}{\pi}\sigma_y\tan^{-1}\sqrt{\frac{\ell}{-x}}\quad\text{for } (x<0). \tag{21}$$

Numerical estimate of Eqs. (18)–(21) are given in Fig. 3 for $\ell/L=0.1$. When the size of the cohesive zone is much smaller than the slipping region, $\ell/L\ll 1$, these equations can be approximated as

$$\sigma_{\text{st}}(x)=\left(\frac{2}{\pi}\sqrt{\frac{\ell}{L}}\sigma_y\right)\sqrt{\frac{x-L}{x}}\quad\text{for } (x>L), \tag{22}$$

$$U'(x)=2\frac{\sigma_y}{\pi\mu\beta}\sqrt{\frac{\ell}{x}}\left(1-\frac{x/L}{1+\sqrt{1-x/L}}\right)\quad\text{for } (\ell\ll x<L), \tag{23}$$

$$\sigma^{(-)}(x)=\frac{2}{\pi}\sigma_y\left[\tan^{-1}\sqrt{\frac{\ell}{-x}}+\sqrt{\frac{\ell}{-x}}(\sqrt{1-x/L}-1)\right]\quad\text{for } (x<0). \tag{24}$$

These coincide to the ‘‘kinematic’’ analysis of stress for the self-healing crack by Freund [2].

The solution contains the three unknown parameters, namely, v, ℓ, and L, and the one external parameter σ_∞, which is directly related with Δ by Eq. (4). For the physical conditions to determine the parameters, we have the nondivergent condition of stress (10) and the matching condition for the cohesive zone at the crack tip (13). In the $\ell/L\ll 1$ case, these conditions give

$$\frac{\sigma_\infty}{\sigma_y}=\frac{2}{\pi}\sqrt{\frac{\ell}{L}},\quad \ell=\frac{\pi}{2}\beta\mu\frac{\delta}{\sigma_y}, \tag{25}$$

from which we have

$$L=\frac{2}{\pi}\sigma_y\beta\mu\delta\frac{1}{\sigma_\infty^2}. \tag{26}$$

The total displacement D is given by $U(L)$; in the $\ell/L\ll 1$ case, from the expression (23), we have

$$D=\frac{\pi}{2}\frac{\sigma_\infty}{\mu\beta}L, \tag{27}$$

and from Eq. (26), this can be shown to satisfy the relation

$$D\sigma_\infty=\sigma_y\delta\equiv\Gamma. \tag{28}$$

This equation simply represents that the elastic energy released by the displacement D equals to the fracture energy Γ, which implies the solution represents the fracture propagating without emitting any sound.

There are some points that need comments about this solution.

(1) In the case of non-healing solution, the physical conditions to determine the parameters v and ℓ are the nondivergence condition at the crack tip and the matching condition at the end of the cohesive zone. The former condition reduces to the energy criterion by Griffith [11], i.e., the crack criterion given by the simple phenomenological parameter or the fracture energy.

On the other hand, in the case of the self-healing pulse solution, the solution contains the additional parameter, the pulse length L. The additional condition should come from the resticking condition. Difference from the condition at the breaking point is that the solution does not have a stress divergence at the resticking point for any L, therefore we cannot impose the nondivergence condition there to determine it. This implies that the resticking condition cannot be expressed by a simple macroscopic quantities like the fracture energy, but depends upon microscopic quantities such as resticking stress or critical slipping speed for resticking.

(2) The solution is obtained in the $\alpha\to 0$ limit, or the infinite system width limit, therefore the situation represented by the present solution is that the total displacement D is much smaller than the externally imposed displacement Δ; only negligible part of the total stress is released by the slip. In real earthquakes, it has been estimated [1] that only relatively small fraction ($\sim$10%) of stress is released although there are substantial variations in the released stress from an earthquake event to another. On the other hand, in the simulations for the spring-block system for the earthquake dynamics, the system overshoots in the big events [3–5]. For such cases, we need to solve Eq. (1) for the nonzero α, where the integral kernel in Eqs. (8) and (9) decay exponentially for large $|x-y|$, but their actual form is not simple and I am unable to extract any compact results.

(3) The present solution reduces to Freund's one [2] in the $\ell/L\ll 1$ limit as is shown in Eqs. (22)–(24), but the difference between the present treatment and Freund's is that we define the model as a dynamical one in the sense that the dynamical features can be determined for a given external physical condition; v, ℓ, and L can be calculated as a function of σ_∞ if a resticking condition is given. On the other hand, in Freund's treatment, simply a kinematic solution is given for arbitrary values of parameters v, L, and σ_∞, therefore the dynamical parameters cannot be determined for a given physical situation.

In summary, I obtained the analytic solution for the self-healing pulse of the crack propagation for the simple dynamical model of fracture in a two-dimensional continuum in the infinite width limit. In order to suppress the divergence in the stress, the model needs the cohesive zone at the crack tip, but not at the resticking point. This implies the fracture speed of the solution is determined by the microscopic parameters of resticking condition, such as resticking stress or critical slipping speed for resticking.

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APPENDIX

I present the derivation of Eq. (18) from Eq. (16) in this appendix. Let us start by introducing the complex function $\Phi(z)$

$$\Phi(z)\equiv\int_L^\infty \frac{dy}{\pi}\frac{\sigma_{\rm st}(y)/\sqrt{y}}{y-z}, \tag{A1}$$

which is regular except for $z\geqslant L$ along the real axis if the integral converges. Then we have

$$\frac{\sigma_{\rm st}(x)}{\sqrt{x}}=\frac{1}{2i}[\Phi(x+i\epsilon)-\Phi(x-i\epsilon)], \tag{A2}$$

$$\mathrm{P}\int_L^\infty \frac{dy}{\pi}\frac{\sigma_{\rm st}(y)/\sqrt{y}}{y-x}=\frac{1}{2}[\Phi(x+i\epsilon)+\Phi(x-i\epsilon)] \tag{A3}$$

for $x>L$, where ϵ is the positive infinitesimal. Using Eq. (16), the second equation can be written as

$$\frac{1}{2i}[e^{(\pi/2)i}\Phi(x+i\epsilon)-e^{-(\pi/2)i}\Phi(x-i\epsilon)]$$
$$=\mu\beta\frac{U_0'(x)}{\sqrt{x}}\quad \text{for}\quad x>L. \tag{A4}$$

Now we define another complex function $X(z)\equiv 1/\sqrt{L-z}$ with the branch cut $z\geqslant L$ along the real axis and choose the branch with

$$X(x+i\epsilon)=\frac{1}{\sqrt{x-L}}e^{(\pi/2)i},\quad X(x-i\epsilon)=\frac{1}{\sqrt{x-L}}e^{-(\pi/2)i} \tag{A5}$$

for $x>L$. Then, Eq. (A4) divided by $\sqrt{x-L}$ can be expressed as

$$\frac{1}{2i}[X(x+i\epsilon)\Phi(x+i\epsilon)-X(x-i\epsilon)\Phi(x-i\epsilon)]$$
$$=\mu\beta\frac{U_0'(x)}{\sqrt{x}\sqrt{x-L}}\quad \text{for}\quad x>L. \tag{A6}$$

Since $X(z)\Phi(z)$ is regular except for $z\geqslant L$ along the real axis, we obtain

$$X(z)\Phi(z)=\mu\beta\int_L^\infty\frac{dy}{\pi}\frac{1}{y-z}\frac{U_0'(y)}{\sqrt{y}\sqrt{y-L}}+Q(z), \tag{A7}$$

where $Q(z)$ is a function that is regular except for $z\geqslant L$ and continuous for $z>L$. With this expression and Eq. (A2), we have

$$\sigma_{\rm st}(x)=-\mu\beta\,\mathrm{P}\int_L^\infty\frac{dy}{\pi}\frac{1}{y-x}\sqrt{\frac{x}{y}}\sqrt{\frac{x-L}{y-L}}U_0'(y)$$
$$-Q(x)\sqrt{x}\sqrt{x-L}\quad \text{for}\quad x>L, \tag{A8}$$

but $Q(x)=0$ because $\sigma_{\rm st}(x)$ should be finite for $x>L$ and $\sigma_{\rm st}(x)\to\mathrm{const}$ $(x\to\infty)$.

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